60-74, Contract AF 18(600)-392 (December 1959).

[10] Golik, R. J., "On dissipative mechanisms within separated flow regions (with special consideration to energy transfer across turbulent, compressible, $Pr = 1$, mixing regions," thesis submitted to the Graduate College of Univ. Ill. in partial fulfillment of the requirements for the Ph.D. degree in Mech. Eng., Urbana, Ill. (June 1962).

[11] Battin, W. J., "A proposal for thesis research on transient temperature characteristics of the wake region in non-isoenergetic turbulent jet mixing," submitted to Graduate Committee Chairman, Dept. Mech. Ind. Eng., Univ. Ill. (March 1962).

Free-Convection Laminar Boundary Layers in Oscillatory Flow

Rattan Sagar Nanda* and Vishnoo Prasad Sharma†
Indian Institute of Technology, Kharagpur, India

The study of laminar boundary layers in oscillatory flow with a steady mean was initiated by Lighthill,[1] who considered the effect of fluctuations of freestream velocity on the skin friction and heat transfer for plates and cylinders. Since then various aspects of this problem have been considered by many workers.[2–4] The present note considers the corresponding free-convection problem for a vertical flat plate, when the plate temperature oscillates in time about a constant nonzero mean while the freestream is isothermal. The boundary layer equations, in terms of stream function by which the continuity equation is identically satisfied, are, in dimensionless form,

$$\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} = G + \psi_{yyy} \tag{1}$$

$$G_t + \psi_y G_x - \psi_x G_y = (1/\sigma)G_{yy} \tag{2}$$

where σ is the Prandtl number. The boundary conditions to be satisfied are

$$\begin{aligned} &y = 0 \qquad \psi_y = \psi_y = 0 \\ &G = G_0(1 + \epsilon e^{i\omega t}) \qquad \epsilon \ll 1 \\ &y \to \infty \qquad \psi_y \to 0 \qquad G \to 0 \end{aligned} \tag{3}$$

Now write ψ and G as the sum of steady and small oscillating components:

$$\begin{aligned} \psi &= A(x,y) + \epsilon e^{i\omega t} B(x,y) \\ G &= P(x,y) + \epsilon e^{i\omega t} Q(x,y) \end{aligned} \tag{4}$$

where (A,P) is the steady mean flow and satisfies

$$\begin{aligned} &A_y A_{xy} - A_x A_{yy} = P + A_{yyy} \\ &A_y P_x - A_x P_y = (1/\sigma)P_{yy} \\ &y = 0 \qquad A_x = A_y = 0 \qquad P = G_0 \\ &y \to \infty \qquad A_y \to 0 \qquad P \to 0 \end{aligned} \tag{5}$$

Neglecting squares of ϵ and dividing by $e^{i\omega t}$, one finds that (B,Q) satisfy the following differential set:

$$\begin{aligned} &i\omega B_y + B_y A_{xy} + A_y B_{xy} - A_x B_{yy} - B_x A_{yy} = B_{yyy} + Q \\ &i\omega Q + A_y Q_x + B_y P_x - A_x Q_y - B_x P_y = (1/\sigma)Q_{yy} \\ &y = 0 \qquad B_x = B_y = 0 \qquad Q = G_0 \\ &y \to \infty \qquad B_y \to 0 \qquad Q \to 0 \end{aligned} \tag{6}$$

Received by IAS April 25, 1962.

* Lecturer, Department of Mathematics.

† Associate Lecturer, Department of Mathematics.

Method of Solution

Considering set (5), one finds that this is the boundary layer problem of steady free-convection flow over a vertical plate and can be reduced to ordinary differential equations by the similarity transformation

$$\begin{aligned} \eta &= y(G_0/x)^{1/4} \\ A &= 4(G_0 x^3)^{1/4} F(\eta) \\ P &= G_0\theta(\eta) \end{aligned} \tag{7}$$

Set (6) is considered next. It is convenient to write B and Q as sums of in-phase and out-of-phase components. Substitute $B = M + iN$, $Q = R + iS$ in Eq. (6) and separate real and imaginary parts to get

$$\begin{aligned} &-\omega N_y + M_y A_{xy} + A_y M_{xy} - A_x M_{yy} - M_x A_{yy} = M_{yyy} + R \\ &-\omega S + A_y R_x + M_y P_x - A_x R_y - M_x P_y = (1/\sigma)R_{yy} \\ &y = 0 \qquad M_x = M_y = 0 \qquad R = G_0 \\ &y \to \infty \qquad M_y \to 0 \qquad R \to 0 \end{aligned}$$

and

$$\begin{aligned} &\omega M_y + N_y A_{xy} + A_y N_{xy} - A_x N_{yy} - N_x A_{yy} = N_{yyy} + S \\ &\omega R + A_y S_x + N_y P_x - A_x S_y - P_y N_x = (1/\sigma)S_{yy} \\ &y = 0 \qquad N_x = N_y = S = 0 \\ &y \to \infty \qquad N_y \to 0 \qquad S \to 0 \end{aligned} \tag{8}$$

Low-Frequency Oscillations

Similarity solutions of the partial differential set (8), as found in the case of set (5), do not exist. However, for low-frequency oscillations the series expansions

$$M = \left(\frac{x_1}{\omega^2}\right)^{3/4} \sum_{p=1}^{\infty} M_p(\eta) x_1^{p-1}$$

$$N = \left(\frac{x_1^5}{\omega^6}\right)^{1/4} \sum_{p=1}^{\infty} N_p x_1^{p-1}$$

$$R = \sum_{p=1}^{\infty} R_p(\eta) x_1^{p-1}$$

$$S = (x_1)^{1/2} \sum_{p=1}^{\infty} S_p(\eta) x_1^{p-1}$$

where $x_1 = (x\omega^2/G_0)$ may be introduced. Substituting in (8) and equating powers of x_1, one obtains the following set of ordinary equations:

$$\begin{aligned} &M_1''' + 3FM_1'' - 4F'M_1' + 3F''M_1 = -R_1 \\ &(1/\sigma)R_1'' + 3FR_1' = -\tfrac{3}{4}\theta' M_1 \\ &N_1''' + 3FN_1'' - 6F'N_1' + 5F''N_1 = M_1' - S_1 \\ &(1/\sigma)S_1'' + 3FS_1' - 2F'S_1 = R_1 - \tfrac{5}{4}\theta' N_1 \end{aligned} \tag{9}$$

$$\begin{aligned} &M_n''' + 3FM_n'' - 4nF'M_n' + (4n-1)F''M_n = -R_n - N_n' \\ &(1/\sigma)R_n'' + 3FR_n' - 4(n-1)F'R_n = -S_{n-1} - (n - \tfrac{1}{4})\theta' M_n \\ &N_n''' + 3FN_n'' - (4n+2)F'N_n' + (4n+1)F''N_n = M_n' - S_n \\ &(1/\sigma)S_n'' + 3FS_n' - (4n-2)F'S_n = R_n - (n + \tfrac{1}{4})\theta' N_n \end{aligned} \tag{10}$$

where

$$n = 2, 3, 4, \ldots$$

High-Frequency Oscillation

If the frequency of oscillation is high enough, Eqs. (6) reduce to

$$i\omega B_y = B_{yy} + Q$$
$$i\omega Q = (1/\sigma)Q_{yy}$$

from which one easily obtains

$$Q = G_0 e^{-(i\omega\sigma)^{1/2}y}$$
$$B = [G_0/i\omega(\sigma - 1)][e^{-(i\omega)^{1/2}y} - e^{-(i\omega\sigma)^{1/2}y}]$$

which is of the "shear-wave" type, predicting a phase advance of 45° in the local rate-of-heat-transfer fluctuations and an equivalent phase lag in the skin-friction oscillations.

For sufficiently small values of ω, only the first term of the series expansion will be significantly important. It easily is verified that $M_1 = F + \eta F'$ and $R_1 = \theta + \frac{1}{4}\eta\theta'$. It then remains to determine N_1 and S_1. As a preliminary step, N_1 and S_1 were determined using the von Kármán-Pohlhausen method.[5] The results indicate that there exists a critical frequency ω_0 such that $x_1 = x\omega^2/G_0 = 0.7$, which separates the regions of applicability of low- and high-frequency solutions. However, to predict the results more accurately, these equations are being integrated numerically, and the results will be presented in a separate paper.

References

[1] Lighthill, M. J., "The response of laminar skin friction and heat transfer to fluctuations in the stream velocity," Proc. Roy. Soc. (London) **A224**, 1 (1954).

[2] Hill, P. G., "Laminar boundary layers in oscillatory flow," Gas Turbine Lab. Rept. 45 (April 1958).

[3] Rosenzweig, M. L., "Response of laminar boundary layer to impulsive motions," Cornell Univ. Rept. TN 59–92 (1952).

[4] Rott, N. and Rosenzweig, M. L., "On the response of the laminar boundary layer to small fluctuations of the free stream velocity," J. Aerospace Sci. **27**, 741–748 (1960).

[5] Nanda, R. S. and Sharma, V. P., "Free convection laminar boundary layers in oscillatory flow," J. Fluid Mech. (to be published).

Structural Damping

R. H. Scanlan* and A. Mendelson†

Case Institute of Technology, Cleveland, Ohio

ARE certain mathematical models of structural damping physically unrealizable? It is probable that true damping mechanisms in structures are of a quite complicated character. However, in a good many practical cases, it has appeared possible to account in a reasonable measure for the damping in an overall way by means of a linear model. It goes without saying that this model must be free of gross anomalies; a poor model may confuse the situation. When the simplified model has been chosen, nothing other than an approximation to the overall effect of the damping is expressed concerning the true damping mechanism. To inquire, then, into whether one or another simplified model itself is physically realizable or unrealizable would appear to be a less rewarding side of the question. Crandall, in a recent publication[1] raises the question of the physical realizability of one of the most familiar linear damping models; the present note discusses this model.

Received by IAS November 13, 1962.

* Professor of Engineering Mechanics, Engineering Division.

† On leave from NASA Lewis Research Center, Cleveland, Ohio.

The damping model in question was employed in 1938 by Theodorsen and Garrick[2] in early flutter studies, and they ascribed it to Becker and Föppl. Specifically, it employs the device of introducing linear structural damping into the typical flutter equations having complex coefficients, not as a viscous velocity term, but as a term $ig\omega_n^2 x$, where $i = (-1)^{1/2}$, $g > 0$ is a damping coefficient, and x is displacement of a typical degree of freedom, the natural circular frequency of which is ω_n. This device has the well-known effect of creating a damping term in phase with velocity $\dot{x}$ and proportional to a displacement x which has the form $e^{i\omega t}$ with $\omega > 0$, since the term is advanced 90° in phase by the factor i. Note that, in this sense, since the usual flutter analysis of the type alluded to was made precisely for nondecaying sinusoidal oscillations of the form $e^{i\omega t}$ only, the device is effective in its intended purpose within this context. Therein, incidentally, no occasion arises to consider ω other than positive.

As an equation typical of this situation, consider the following:

$$\ddot{x} + (1 + ig)\omega_n^2 x = Ae^{i\omega t} \tag{1}$$

where ω_n is the natural circular frequency of the undamped system and $\omega > 0$ is the forcing circular frequency, A being some complex constant.

Briefly recall here the solution of Eq. (1): it consists of the free vibration (solution for $A = 0$) plus a forced vibration at circular frequency ω. Crandall[1] raises the question of the physical unrealizability of the solution of Eq. (1) "if . . . negative frequencies are to be considered . . . ." Normally, as was pointed out, only positive ω is considered in the forced vibration. "Negative frequencies" are to be considered in Eq. (1), therefore, only in the cases where the situation described by Eq. (1) is extended to other meanings than the one originally intended (such as modification to greater generality of the right-hand side and, in particular, an extension into the negative Fourier domain) or in the homogeneous case, i.e., the free vibration. In examining the free vibration an anomaly is, in fact, encountered. It is found that, for *arbitrary* initial conditions, and for either $g > 0$ *or* $g < 0$, *no decaying solution exists*, but rather there *always* exists a portion of the solution having exponentially increasing amplitude with time. This is demonstrated as follows: Let g be positive *or* negative. Take $A = 0$ in Eq. (1) and assume a solution in the form

$$x = x_0 e^{i\omega_d t}$$

with $\omega_d = \omega_r + i\omega_i$ (ω_r, ω_i real). Use of this solution in Eq. (1) yields

$$-\omega_d^2 + \omega_n^2(1 + ig) = 0$$

which in turn gives the following solutions for ω_r and ω_i:

$$\omega_r = \pm\, \omega_n \left[\frac{1 + (1 + g^2)^{1/2}}{2}\right]^{1/2}$$

$$\omega_i = \pm\, \omega_n \frac{g}{\{2[1 + (1 + g^2)^{1/2}]\}^{1/2}}$$

The solution for x is then

$$x = x_{01}e^{i\omega_1 t} + x_{02}e^{i\omega_2 t}$$

with x_{01} and x_{02} arbitrary constants and

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = \pm\, \omega_n \left\{\left[\frac{1 + (1 + g^2)^{1/2}}{2}\right]^{1/2} + i\,\frac{g}{\{2[1 + (1 + g^2)^{1/2}]\}^{1/2}}\right\}$$

If one insists that the solution must represent a decaying motion for arbitrary initial conditions (x_{01}, $x_{02} \pm 0$), the sign